

# NEW APPROACH TO INDUCED QCD

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## Abstract

Matrix model approach to multicolor induced QCD based on the quenched momentum prescription is presented. It is shown that this model exhibits the reduction of spatial degrees of freedom: the partition function is determined by the solution of one dimensional quantum mechanical problem while the D-dimensional scalar field correlators coincide with the same type correlators in the two-dimensional induced QCD.

1. Investigation of the large number of colors  $N_c$  limit of QCD permits a deeper insight into the nature of the strong interaction. Analytical summation of planar diagrams leads to a kind of masterfield equation. The considerable reduction of actual degrees of freedom occurring when  $N_c \rightarrow \infty$  underlies this equation. The large number of "angular-type" variables (unitary transformations) is effectively "integrated out" in matrix models. The remaining variables form a meanfield described in terms of the masterfield equation.

The case of induced QCD will be considered here. This model was introduced by Kazakov and Migdal four years ago [1]. Here the induced QCD is revised but instead of the lattice version of QCD in Kazakov-Migdal model (KMM) the prescription proposed by Gross and Kitazawa [2] for the planar Feynman diagrams is adopted. As in the KMM the absence of the gluon kinetic energy enables to integrate over the unitary matrix in closed form. However contrary to the KMM case the explicit expression for the "angular" integrals are not used in the present approach since it entirely avoids the introduction of mean field as well as the solution of the masterfield equation. The main goal of it is to establish the connection between 4D induced QCD and the same lower dimensional theory.

2. Since the Gross and Kitazawa prescription plays the crucial role in the following treatment it is instructive to begin with brief reminding of it.

The quantum scalar field in adjoint representation  $\varphi(x)$  is replaced by the  $N_c \times N_c$  matrix  $\varphi$  which does not depend on the  $x_\mu$  variables. The field derivative is replaced by the commutator

$$\partial_\mu \varphi \rightarrow i [P_\mu, \varphi],$$

where  $P_\mu$  are diagonal  $N_c \times N_c$  matrixes:

$$(P_\mu)^{ij} = \delta^{ij} p_\mu^i. \quad (1)$$

The functional (euclidean) integral

$$Z = \int \prod_x D\varphi(x) \exp \left\{ \int d^D x \text{Tr} \left( -\frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right) \right\} \quad (2)$$

turns into the matrix integral

$$Z = \int d\varphi \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ \text{Tr} \left( \frac{1}{2} [P_\mu, \varphi]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right) \right\} \quad (3)$$

where

$$d\varphi = \prod_{ij} d\varphi_{ij}.$$

When  $N_c \rightarrow \infty$  the diagrams generated by the perturbative expansion of the second integral coincide with the planar Feynman diagrams for the first one,  $\Lambda$  being the ultraviolet cutoff. Indeed the matrix model propagator is

$$\langle \varphi_{ij} \varphi_{sp} \rangle = \frac{\delta_{ip} \delta_{js}}{(p_\mu^i - p_\mu^j)^2 + M^2}. \quad (4)$$

Drawing the Feynman propagator by the doubleline one can assign to each line its own momentum  $p_1$  or  $p_2$  so that the total momentum flowing through the propagator is  $k = p_1 - p_2$ . After this substitution the loop integrals turn into the integrals over the momenta  $p_{1,2}$ . For planar graphs the momentum loops are identical to the color ones inside which the indexes  $i, j$  in (4) circulate. If the D-dimensional hypercube in the momentum space is divided into  $N_c$  equal cells with the volume  $\Lambda^D/N_c$  and the vectors  $p_\mu^i$  in (4) are choosen to lay inside these cells the sums over the color indexes in the matrix diagrams for the model (3) yield the integral sums for the momentum loop integrals in the planar Feynman diagrams for the functional integral (2). When  $N_c \rightarrow \infty$  and  $\Lambda$  is fixed the sums turn into the integrals. After this the limit  $\Lambda \rightarrow \infty$  can be taken. The hermitean  $N_c \times N_c$  matrix  $P_\mu$  (1) plays the role of the quenched momentum and the components  $p_\mu^i$  are the values it takes.<sup>1</sup>

The unitary matrixes  $e^{iP x}$  can be treated as a finite dimension approximation to the space shift operator. One can define

$$\varphi(x) \equiv e^{iP x} \varphi e^{-iP x}.$$

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<sup>1</sup>It is really not necessary to identify the number of the terms in the integral momentum sum with  $N_c$ . Indeed, while a planar Feynman diagram contribution is  $N_c^2 G^F$  where  $G^F$  does not depend on  $N_c$ , the rank  $N$  matrix model gives for a planar graph  $G_{planar} = N^2 \overline{G}$ . The quenched momentum prescription ensures  $\overline{G} \rightarrow G^F$  for  $N \rightarrow \infty$  and it is this property that makes the matrix model to be equivalent to the field theory. Thus only the planarity is important here.

Then the correlator

$$\langle \text{Tr} \varphi(x_1) \cdots \varphi(x_n) \rangle$$

calculated in the matrix model (3) in the leading  $N_c$  order will be the same as in the scalar field theory (2).

The incorporation of gauge field in this prescription is straightforward:

$$\begin{aligned} A_\mu(x) &\rightarrow A_\mu \\ \frac{1}{i} D_\mu &\rightarrow P_\mu + \frac{g}{\sqrt{N_c}} A_\mu \\ i \frac{g}{\sqrt{N_c}} G_{\mu\nu} &= [D_\mu, D_\nu] \end{aligned}$$

Here  $A_\mu$  is a  $N_c \times N_c$  hermitean matrix,  $D_\mu$  is a covariant derivative and the commutator in the last formula is understood in the matrix sense. The (euclidean) matrix action can be written as

$$S = -\frac{1}{4} \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr} G_{\mu\nu}^2 + S_{\text{gf}}$$

where  $S_{\text{gf}}$  is an appropriate gauge-fixing term. The matrix transformation

$$\frac{g}{\sqrt{N_c}} A_\mu \rightarrow V^{-1} A_\mu V + \frac{g}{\sqrt{N_c}} V^{-1} [P_\mu, V]$$

where  $V$  is a unitary matrix is equivalent through the relations  $A_\mu(x) = e^{iPx} A_\mu e^{-iPx}$ ,  $V(x) = e^{iPx} V e^{-iPx}$  to a local gauge transformation in the field theory.

Very important for the following is the additional constraint imposed on the measure of the matrix integral, namely, the integration is carried out only over the matrixes  $A_\mu$  for which the covariant derivative  $D_\mu$  has the same eigenvalues as the matrix  $P_\mu$ . An equivalent form of this restriction is to rewrite the functional integral as

$$Z = \int \prod_\mu dA_\mu c_\mu e^{-S}$$

where

$$c_\mu \sim \int dV_\mu \delta\left(\frac{1}{i} D_\mu - V_\mu^{-1} P_\mu V_\mu\right) \quad (5)$$

and  $dV_\mu$  denotes the invariant measure on the  $SU(N_c)$  group. If it were not for the constraint (5) the  $P_\mu$  matrixes would be completely excluded from the integral by shifting matrix variables  $\frac{g}{\sqrt{N_c}} A_\mu \rightarrow P_\mu + \frac{g}{\sqrt{N_c}} A_\mu = \frac{1}{i} D_\mu$ . The constraints (5) have a rather obvious meaning: any component  $A_\mu$  of the gluon field can be set to zero by the gauge transformation  $V_\mu$  special for each  $\mu$  (only for the pure gauge field there is a matrix  $V$  common to all  $\mu$ ).

**3.** The induced QCD lagrangian describes the theory containing the gluons and the adjoint scalars

$$\mathcal{L} = \frac{1}{2}Tr(D_\mu\varphi)^2 - \frac{1}{2}TrM^2\varphi^2 \quad (6)$$

but without the pure gluon term. The integration over the scalar field yields the functional determinant

$$Det(M^2 - D^2)^{-\frac{1}{2}} = \exp \left\{ -\frac{1}{96\pi^2} g^2 N_c \log \frac{\Lambda}{M} \int d^4x Tr G_{\mu\nu}^2 + O(1/M) \right\}.$$

Here  $O(1/M)$  denotes the terms finite when the ultraviolet cutoff  $\Lambda \rightarrow \infty$ . They are suppressed by the powers of the scalar mass  $M$  which is assumed to be very large (although  $M \ll \Lambda$ ). The theory (6) seems to be similar in the limit  $M \rightarrow \infty$  ( $M \ll \Lambda$ ) to the gluodynamic,  $g^2 \sim 1/\log \Lambda/M$  being the coupling constant. One can consider a more general case by adding to the lagrangian (6) the scalar field interaction  $V(\varphi) = \sum v_n \varphi^n$ .

According to the quenched momentum prescription the matrix model integral for this theory in the hamiltonian gauge  $A_0 = 0$  reads

$$\begin{aligned} Z &= \int \prod_\mu dA_\mu c_\mu \delta(A_0) d\varphi \\ &\times \exp \left( \frac{2\pi}{\Lambda} \right)^D Tr \left\{ \frac{1}{2} \sum_\mu [D_\mu, \varphi]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right\}. \end{aligned}$$

Resolving  $\delta$ -functions in the constraints  $c_\mu$  gives

$$\begin{aligned} Z &= \int \prod_\mu dV_\mu d\varphi \delta \left( \frac{N_c}{g} (P_0 - V_0^{-1} P_0 V_0) \right) \\ &\times \exp \left\{ \left( \frac{2\pi}{\Lambda} \right)^D Tr \left( \frac{1}{2} \sum_\mu [V_\mu^{-1} P_\mu V_\mu, \varphi]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right) \right\}. \end{aligned}$$

Writing here the scalar field matrix as  $\varphi = U^{-1} \varphi_d U$  where  $U$  is a unitary matrix and  $\varphi_d$  is a diagonal one the integral takes a form

$$\begin{aligned} Z &= \int \prod_\mu dV_\mu dU d\varphi_d \Delta^2(\varphi_d) \delta \left( \frac{N_c}{g} (P_0 - V_0^{-1} P_0 V_0) \right) \\ &\times \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ \frac{1}{2} \sum_\mu Tr [P_\mu, V_\mu U^{-1} \varphi_d U V_\mu^{-1}]^2 \right. \\ &\quad \left. - \frac{1}{2} M^2 \varphi_d^2 - V(\varphi_d) \right\}, \end{aligned}$$

$V(\varphi_d) \equiv \sum_i V(\varphi_i)$  is the sum over the eigenvalues,  $\Delta(\varphi_d)$  is the Van der Monde determinant. Changing the integration variables

$$\begin{aligned} V_\mu U^{-1} &= \tilde{V}_\mu, & \mu = 1, \dots, D-1, \\ V_0 U^{-1} &= \tilde{U}^{-1} \end{aligned} \quad (7)$$

allows to separate out the gauge-fixing term:

$$\begin{aligned}
Z &= \int dV_0 \delta \left( \frac{N_c}{g} (P_0 - V_0^{-1} P_0 V_0) \right) \cdot \int d\tilde{U} \prod_{\mu=1}^{D-1} d\tilde{V}_\mu d\varphi_d \Delta^2(\varphi_d) \\
&\times \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ \frac{1}{2} \text{Tr}[P_0, \tilde{U}^{-1} \varphi_d \tilde{U}]^2 + \frac{1}{2} \text{Tr} \sum_{\mu=1}^{D-1} [P_\mu, \tilde{V}_\mu \varphi_d \tilde{V}_\mu^{-1}]^2 \right. \\
&\quad \left. - \frac{1}{2} M^2 \varphi_d^2 - V(\varphi_d) \right\}.
\end{aligned}$$

Dropping the tildes and redefining  $U_0 = V_0$  the partition function can be rewritten as ( $a \sim 1/\Delta^2(P_0)$  is the normalization constant)

$$\begin{aligned}
Z &= a \int d\varphi_d \Delta^2(\varphi_d) \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ -\frac{1}{2} M^2 \varphi_d^2 - V(\varphi_d) \right\} \\
&\times \int \prod_{\mu}^D dV_\mu \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ \frac{1}{2} \sum_{\mu=0}^{D-1} \text{Tr}[P_\mu, V_\mu \varphi_d V_\mu^{-1}]^2 \right\}. \quad (8)
\end{aligned}$$

The integration over the unitary matrixes in (8) decays into the product of  $D$  equal integrals (they are equal since the matrixes  $P_\mu$  are unitary equivalent).

Consider the integral

$$Z_\varphi = \int dV \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ \frac{1}{2} \text{Tr}[P_\mu, V \varphi_d V^{-1}]^2 \right\} \quad (9)$$

for the fixed matrix  $\varphi_d$  and index  $\mu$ . An explicit calculation of this expression is a rather nontrivial problem although an essential progress has been made recently in studying the similar type integrals [3]. Instead calculating it the much more weak "scaling" property of this integral will be enough for the following.

The main contribution to the integral over the matrixes  $\varphi$  in (9) comes from the domain where  $\varphi_{ij} \sim 1$  and where the typical scale of the eigenvalues  $\varphi_d \sim N_c^{1/2}$ . It is just the region that gives the leading contribution of the order  $N_c^2$  to the free energy while the measure of the rest integration domain tends to zero in the  $N_c \rightarrow \infty$  limit. In this limit the eigenvalues distribution is described by the smooth density function  $\rho(\lambda)$ .

Since  $\text{Tr}[P_\mu, \varphi]^2 \sim N_c^2$  one can expect

$$Z_\varphi = \exp \left\{ N_c^2 F + O(N_c) \right\} \quad (10)$$

where the coefficient  $F$  depends on the eigenvalues of the matrix  $\varphi$  and for the large  $N_c$  can be treated as a functional  $F = F[\rho]$ . Eq.(10) is a consequence of the classical nature of the large  $N_c$  limit which manifests itself in the factorization of the correlators of colorless ( $U(N_c)$  invariant) operators [4].

The large  $N_c$  behavior of the unitary matrix integral (9) looks like as if it is dominated by a saddle point. There is a representation which makes the  $N_c^2$  dependence in the integrand (9) to be explicit [5]. In the large  $N_c$  limit  $SU(N_c)$  algebra is equivalent to the infinite dimensional Lie algebra of area preserving diffeomorphisms of the sphere  $SDiff(S^2)$ . The matrix  $A$  from  $SU(N_c)$  algebra transforms in this representation into the function  $A(x_1, x_2, x_3)$  on the unit sphere  $x_1^2 + x_2^2 + x_3^2 = 1$ , the commutator being replaced by Poisson bracket

$$\lim_{N_c \rightarrow \infty} \frac{N_c}{i} [A, B] = \{A, B\}$$

which are defined as

$$\{A, B\} = x_i \varepsilon_{ikl} \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial x_l}.$$

The integral (9) turns into the functional (infinite order) integral over the group of area preserving diffeomorphisms

$$Z_\varphi = \int \mathcal{D}V \exp \left[ \frac{1}{2} N_c^2 \left( \frac{2\pi}{\Lambda} \right)^D \int d\Omega \{p, V(\varphi)\}^2 \right] \quad (11)$$

where  $V(\varphi)$  denotes the action of the diffeomorphism on the function  $\varphi$  and  $\int d\Omega$  is the integral over the unit sphere. The functions  $\varphi$  and  $p$  correspond to the matrixes  $\varphi$  and  $P_\mu$  in the integral (9). Their particular structure as well as the result of the action of  $V$  on  $\varphi$  is not important here. Only one thing is significant for the following, namely, comparing the expression (11) with (10) one can conclude that

$$\int dV \exp \left\{ -\frac{1}{2} \alpha \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr}[P, V^{-1} \varphi_D V]^2 \right\} = \exp \{ \alpha N_c^2 F + O(N_c) \}$$

since it is nothing more than rescaling of  $N_c$  by the factor  $\sqrt{\alpha}$ . Thus the power of the integral  $Z_\varphi$  can be replaced within the leading order accuracy by the single integral

$$(Z_\varphi)^D = \int dV \exp \left\{ -\frac{1}{2} D \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr}[P, V^{-1} \varphi_D V]^2 \right\}. \quad (12)$$

The "scaling property" (12) will be the central point for the following treatment. All other results are more or less trivial consequences of it.

The relation (12) applied to the integral (8) gives

$$\begin{aligned} Z &= a \int d\varphi_d \Delta^2(\varphi_d) dV \\ &\times \exp \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr} \left\{ \frac{1}{2} D [P_\mu, V \varphi_d V^{-1}]^2 - \frac{1}{2} M^2 \varphi_d^2 - V(\varphi_d) \right\}. \end{aligned}$$

After combining the factors in the integration measure the partition function (8) takes the final form

$$Z = a \int d\varphi \exp \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr} \left\{ \frac{1}{2} D [P_1, \varphi_d]^2 - \frac{1}{2} M^2 \varphi_d^2 - V(\varphi_d) \right\}. \quad (13)$$

Here the matrix  $P_\mu$  index is fixed by the value  $\mu = 1$ , the result being clearly independent on the particular choice of the space direction.

The expression (13) is a Gauss integral for  $V(\varphi) = 0$ . In the leading  $N_c$  order

$$\begin{aligned} \log Z &= \text{const} + DN_c^2 \log \Lambda - \frac{1}{2} \sum_{ik} \log [D(p_i^1 - p_k^1)^2 + M^2] = \\ &= \text{const} + DN_c^2 \log \Lambda - \frac{1}{2} N_c^2 V_D \left( \frac{\Lambda}{2\pi} \right)^{D-1} \int_{-\Lambda/2}^{\Lambda/2} \frac{dp}{2\pi} \log(Dp^2 + M^2) \end{aligned}$$

where  $V_D$  is a total space volume<sup>2</sup>. Collecting the terms independent of  $M$  in the factor  $\varepsilon_0$  one gets for the vacuum energy density

$$\varepsilon = \varepsilon_0 + \left( \frac{\Lambda}{2\pi} \right)^D \left\{ \log \left( 1 + \frac{M^2}{D\Lambda^2} \right) + \frac{2}{\sqrt{D}} \frac{M}{\Lambda} \arctan \sqrt{D} \frac{M}{\Lambda} \right\}. \quad (14)$$

4. When the scalar interaction  $V(\varphi) \neq 0$  the evaluation of the integral (13) is equivalent to solving the quantum mechanical problem. Indeed, as is seen from inverting the quenched momentum prescription, it coincides in the large  $N_c$  limit with the partition function

$$Z_D = \int \prod_x \mathcal{D}\varphi(x) \exp \text{Tr} \int d^D x \left\{ -\frac{1}{2} D (\partial_1 \varphi)^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right\} \quad (15)$$

Since there is only one actual degree of freedom contributing to (15)

$$Z_D = \exp \left\{ -V_D \left( \frac{\Lambda}{2\pi} \right)^{D-1} E \right\}$$

where  $E$  is the ground state energy for the one-dimensional system described by the integral

$$Z_1 = \int \prod_t \mathcal{D}Q(t) \exp N_c \text{Tr} \int dt \left\{ -\frac{1}{2} D \mu \dot{Q}^2 - \frac{1}{2} \mu M^2 Q^2 - V(Q) \right\} \quad (16)$$

in which  $\mu = \Lambda/2\pi$  and  $V(q) = \sum_{n \geq 3} \mu^{1+D(n/2-1)} v_n q^n$ . This relation immediately follows from the lattice version of the theory (15) which is equivalent to the ultraviolet regularization with the cutoff  $\Lambda$ ,  $1/\mu$  being the lattice spacing.

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<sup>2</sup>The value  $V_{\text{tot}} = (2\pi)^D N_c / \Lambda^D$  is to be taken as a space volume in the quenched model prescription to recover the planar graphs. Taking into account the proper order of limits (first  $N_c \rightarrow \infty$  and then  $\Lambda \rightarrow \infty$ ) one can put  $V_{\text{tot}} = N_c V_D$

The system (16) is solved by reduction to the free fermions moving in the external potential  $V(q)$  [6]. The energy is  $E = \sum_{i=1}^{N_c} \mathcal{E}_i$ , where  $\mathcal{E}_i$  are the energies of the lowest  $N_c$  occupied states:

$$\left[ -\frac{1}{2\mu D} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \mu M^2 x^2 + V(x) \right] \psi_i(x) = \mathcal{E}_i \psi_i(x).$$

In the quasiclassical approximation which validity is justified by large  $N_c$

$$E = N_c^2 \int \frac{dp dq}{2\pi} H(p, q) \theta(\mathcal{E} - H(p, q)) \quad (17)$$

where  $\theta$  is the step function,

$$H(p, q) = \frac{p^2}{2D} + \frac{1}{2} M^2 q^2 + \sum_{n \geq 3} \mu^{(D-1)(\frac{n}{2}-1)} v_n q^n$$

and Fermi-level  $\mathcal{E}$  is determined by the equation

$$\int \frac{dp dq}{2\pi} \theta(\mathcal{E} - H(p, q)) = 1.$$

Note that  $E = 0$  for  $M = 0$  and  $V(q) = 0$ .

**5.** Consider now the two-point correlator of the scalar fields:

$$\begin{aligned} K^{(D)}(x) &= \frac{1}{Z} \int \prod_{\mu=0}^{D-1} dA_\mu c_\mu d\varphi \text{Tr}[\varphi(x)\varphi(0)] \delta(A_0) \\ &\times \exp\left(\frac{2\pi}{\Lambda}\right)^D \text{Tr}\left\{\frac{1}{2} \sum_\mu [D_\mu, \varphi]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi)\right\}. \end{aligned} \quad (18)$$

The variables changing (7) brings the integral to the form

$$\begin{aligned} K^{(D)}(x) &= \frac{1}{Z} \int dV_0 \prod_{\mu=1}^{D-1} d\tilde{V}_\mu dU d\varphi_d \Delta^2(\varphi_d) \delta\left[\frac{N_c}{g} (P_0 - V_0^{-1} P_0 V_0)\right] \\ &\times \text{Tr}[e^{iPx} U^{-1} \varphi_d U e^{-iPx} \cdot U^{-1} \varphi_d U] \\ &\times \exp\left(\frac{2\pi}{\Lambda}\right)^D \text{Tr}\left\{\frac{1}{2} [P_0, V_0 U^{-1} \varphi U V_0^{-1}]^2 \right. \\ &\quad \left. + \frac{1}{2} \sum_{\mu=1}^{D-1} [P_\mu, \tilde{V}_\mu \varphi_d \tilde{V}_\mu^{-1}]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi)\right\}. \end{aligned} \quad (19)$$

In contrast to the partition function case the pre-exponential factor prevents to factorize the gauge-fixing term with the integral over  $V_0$ . Nevertheless the



”dimensional scaling property” (12) can be used again to reduce the product of the independent integrals  $Z_\varphi^{D-1}$  to the single one:

$$\begin{aligned}
K^{(D)}(x) &= \frac{1}{Z} \int dV_0 d\tilde{V}_1 dU d\varphi_d \Delta^2(\varphi_d) \delta \left[ \frac{N_c}{g} (P_0 - V_0^{-1} P_0 V_0) \right] \\
&\times \text{Tr}[e^{iPx} U^{-1} \varphi_d U e^{-iPx} \cdot U^{-1} \varphi_d U] \\
&\times \exp \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr} \left\{ \frac{1}{2} [P_0, V_0 U^{-1} \varphi U V_0^{-1}]^2 \right. \\
&\quad \left. + \frac{1}{2} (D-1) [P_1, \tilde{V}_1 \varphi_d \tilde{V}_1^{-1}]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right\}.
\end{aligned}$$

Now the steps leading from (18) to (19) can be repeated in the inverse order which yeilds

$$\begin{aligned}
K^{(D)}(x) &= \frac{1}{Z} \int \prod_{\mu=0}^1 dA_\mu c_\mu d\varphi \text{Tr}[\varphi(x) \varphi(0)] \delta(A_0) \\
&\times \exp \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr} \left\{ \frac{1}{2} [D_0, \varphi]^2 + \frac{1}{2} (D-1) [D_1, \varphi]^2 \right. \\
&\quad \left. - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right\}.
\end{aligned}$$

After rescaling

$$P_0 \rightarrow \sqrt{D-1} P_0 \quad (20)$$

$$\varphi \rightarrow \frac{1}{\sqrt{D-1}} \mu^{\frac{D}{2}-1} \varphi \quad (21)$$

the integral takes a more symmetric form <sup>3</sup>

$$\begin{aligned}
K^{(D)}(x) &= \mu^{D-2} \frac{1}{Z} \int \prod_{\mu=0}^1 dA_\mu c_\mu d\varphi \text{Tr}[\varphi(\sqrt{D-1} x_0, x) \varphi(0)] \delta(A_0) \\
&\times \exp \left( \frac{2\pi}{\Lambda} \right)^D \text{Tr} \left\{ \frac{1}{2} \sum_{\mu=0}^1 [D_\mu, \varphi]^2 - \frac{1}{2} \frac{M^2}{D-1} \varphi^2 - \tilde{V}(\varphi) \right\}
\end{aligned}$$

where

$$\tilde{V}(\varphi) = \sum_{n \geq 3} (D-1)^{-\frac{n}{2}} \mu^{(\frac{D}{2}-1)(n-2)} v_n \varphi^n \quad (22)$$

or

$$K^{(D)}(x_0, x_1, 0, \dots, 0) = \mu^{D-2} \frac{1}{D-1} K^{(2)}(\sqrt{D-1} x_1, x_2).$$

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<sup>3</sup>The first line here is the quenched momentum analog of  $x_0 \rightarrow x_0 \sqrt{D-1}$  replacement. Although the integration region is no more a hypercube in the momentum space it is not essential for a ultraviolet convergent 2D theory.

Here  $K^{(2)}$  is the correlator in two-dimensional theory.

Thus the two-point scalar correlator in the  $D$ -dimensional induced QCD coincides with the same (up to rescaling) correlator calculated in the two-dimensional induced QCD. This result clearly holds for any  $n$ -point scalar correlator provided all the points lay in the same two-dimensional plane.

**6.** Unfortunately the above methods based on the "scaling property" (12) are insufficient for gluon correlators. The tensor structure of the vector field correlators in  $D$ -dimensional space is much more rich than for  $D = 2$  and they can not be reduced to a two-dimensional theory. However there is a special kinematic to which a little modification of the property (12) can be applied.

Consider again a two-point correlator

$$d^{(D)}(x) = \langle Tr n_\mu A_\mu(x) n_\nu A_\nu(0) \rangle \quad (23)$$

where  $n$  is an arbitrary vector of the form  $n_\mu = (0, n_1, \dots, n_{D-1})$  (recall the gauge is  $A_0 = 0$ ) normalized as  $n_\mu^2 = (D-1)$ . One can always choose the coordinate axes in such a way that  $n_\mu = (0, 1, \dots, 1)$ . The vector  $x$  in (23) will be assumed to be of the form  $x = (x_0, x_1, 0, \dots, 0)$  in this basis, so  $(xn)^2 = x_1 = x_\perp^2$ .

It is convenient to deal with the generating functional which in the quenched momentum prescription is

$$\begin{aligned} Z_J &= \int \prod_{\mu=0}^{D-1} dA_\mu c_\mu \delta(A_0) d\varphi \\ &\times \exp \left( \frac{2\pi}{\Lambda} \right)^D Tr \left\{ \frac{1}{2} \sum_\mu [D_\mu, \varphi]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right. \\ &\quad \left. + \mu^D J_2 \sum_\mu A_\mu(x) + \mu^D J_1 \sum_\mu A_\mu(0) \right\}. \end{aligned} \quad (24)$$

The correlator is clearly given by differentiating (24) with respect to the matrixes  $J_{1,2}$ :

$$d^{(D)}(x) = Tr \frac{\delta}{\delta J_2} \frac{\delta}{\delta J_1} \cdot Z_J \Big|_{J_{1,2}=0}.$$

After variables shifting and resolving the  $c_\mu$  constraints  $Z_J$  takes the form

$$\begin{aligned} Z_J &= \exp \left\{ \frac{\sqrt{N_c}}{g} Tr(J_2 + J_1) \sum_\mu P_\mu \right\} \\ &\times \exp \left( \frac{2\pi}{\Lambda} \right)^D Tr \left\{ \frac{1}{2} [P_0, \varphi]^2 - \frac{1}{2} M^2 \varphi^2 - V(\varphi) \right\} \\ &\times \int \prod_{\mu=0}^{D-1} dV_\mu \exp \left( \frac{2\pi}{\Lambda} \right)^D \left\{ Tr \frac{1}{2} [V_\mu^{-1} P_\mu V_\mu, \varphi]^2 \right. \end{aligned} \quad (25)$$

$$- \mu^D \frac{\sqrt{N_c}}{g} J_2 e^{iPx} V_\mu^{-1} P_\mu V_\mu e^{-iPx} - \mu^D \frac{\sqrt{N_c}}{g} J_1 V_\mu^{-1} P_\mu V_\mu \Big\}.$$

The internal integral here is again the product of  $(D - 1)$  equal "angular" integrals. Although they differ from  $Z_\varphi$  (9) by additional terms in the exponent these terms are of the order  $N_c^2$  ( $J_{ik} \sim 1$ ) therefore the same type saddle point behavior is natural to be assumed for these integrals too. The continuous representation like (11) can be also written for each of them as

$$\int \mathcal{D}V \exp N_c^2 \left( \frac{2\pi}{\Lambda} \right)^D \int d\Omega \left[ \frac{1}{2} \{ p, V(\varphi) \}^2 + \mu^D J_2(x) V(p) + \mu^D J_1 V(p) \right]$$

where  $V(p)$  denotes as before the action of the area preserving diffeomorphism  $V$  on the continuous image (function on the unit sphere) of the matrix  $P_\mu$ ,  $J_2(x)$  and  $J_1$  are the continuous images of the matrixes  $e^{-iPx} J_2 e^{iPx}$  and  $J_1$  respectively. This formula enables to replace the product of the integrals in (25) by the single integral for the momentum  $P_1$ . Introducing then the field  $A_1$  through the relation

$$V^{-1} P_1 V = \frac{1}{i} D_1 = P_1 + \frac{g}{\sqrt{N_c}} \mu^{\frac{D}{2}-1} A_1$$

and making rescaling (20), (21) the generating functional takes a final form

$$\begin{aligned} Z_J &= N \exp \left\{ \frac{\sqrt{N_c}}{g} \text{Tr}(J_2 + J_1) \sum_\mu P_\mu - (D - 1) P_1 \right\} \\ &\times \int \prod_{\mu=0}^1 dA_\mu c_\mu \delta(A_0) d\varphi \exp \left( \frac{2\pi}{\Lambda} \right)^2 \text{Tr} \left\{ \frac{1}{2} \sum_\mu [D_\mu, \varphi]^2 - \frac{1}{2} \frac{M^2}{D - 1} \varphi^2 \right. \\ &\quad \left. - \tilde{V}(\varphi) (D - 1) \mu^{\frac{D}{2}+1} J_2 A_1(x) - (D - 1) \mu^{\frac{D}{2}+1} J_1 A_1 \right\} \end{aligned}$$

where  $\tilde{V}(\varphi)$  is given by (22) and the factor  $N$  does not depend on  $J_{1,2}$ . It gives for the correlator

$$d^{(D)}(x_0, x_1, 0, \dots, 0) = (D - 1)^2 \mu^{D-2} \langle \text{Tr} A_1(\sqrt{D - 1} x_0, x_1) A_1(0) \rangle_2 \quad (26)$$

where the right hand side means the correlator in the two-dimensional induced QCD.

Note that a two-point gluon correlator in the dimension  $D$

$$d_{\mu\nu}^{(D)}(x) = \langle \text{Tr} A_\mu(x) A_\nu(0) \rangle$$

is determined by two scalar functions

$$d_{\mu\nu}^{(D)}(x) = x_\mu^\perp x_\nu^\perp d_2^{(D)}(x_0, x_\perp^2) + \delta_{\mu\nu}^\perp d_0^{(D)}(x_0, x_\perp^2).$$

The formula (26) implies one relation between them:

$$\begin{aligned} d_2^{(D)}(x_0, x_\perp^2) + d_0^{(D)}(x_0, x_\perp^2) = \\ = (D-1)^2 \mu^{D-2} [d_2^{(2)}(\sqrt{D-1} x_0, x_\perp^2) + d_0^{(2)}(\sqrt{D-1} x_0, x_\perp^2)]. \end{aligned}$$

There is another way to derive this relation. Instead of (23) one can start from the correlator

$$d_{\mu\mu}^{(D)}(x) = \sum_\mu \langle \text{Tr} A_\mu(x) A_\mu(0) \rangle = \frac{\sqrt{N_c}}{g} \langle \text{Tr} [P_\mu P_\nu - D_\mu(x) D_\mu(0)] \rangle$$

and introduce the generating functional

$$\begin{aligned} Z_\sigma &= \langle \exp \{ \sigma \text{Tr} D_\mu(x) D_\mu(0) \} \rangle, \\ d_{\mu\mu}^{(D)}(x) &= \left. \frac{\partial}{\partial \sigma} Z_\sigma \right|_{\sigma=0}. \end{aligned}$$

The integral for  $Z_\sigma$  allows for the dimensional reduction like (24) since the "angular" part of it is again the product of  $(D-1)$  independent equal integrals.

One can combine both these tricks to derive similar relations for higher correlators with the gluon fields. However there is a lot of tensor structures which are not involved in such relations at all. For instance, the triple gluon vertex does not contribute to any of them because of antisymmetry over the index permutations.

**7.** Essential reduction of the space degrees of freedom occurring in the induced QCD is the main result of this paper. All D-dimensional induced QCD theories turn out to be related through the equality (12). The reason for the "scaling property" (12) lies, probably, in the fact that coordinate and momentum operators can be approximated by large  $N_c$  matrixes. Although there are no finite order representation for the Heisenberg algebra one can construct the matrixes  $X_\mu$  and  $P_\nu$  from  $SU(N_c)$  algebra for which  $[X_\mu, P_\nu] = i\delta_{\mu\nu} + O(1/N_c)$  [7]. It explains why space disappears in the induced QCD: the space-time transformation can be mapped for any dimension D into the same  $SU(N_c)$  group. The quenched momentum prescription is a particular choice of such a map. One should stress the point however that the dimensional scaling is valid only for the gauge theories because only for them the "angular" (over unitary matrixes) integrals are carried out separately for each field component. It would be impossible for a scalar theory where the unitary rotation is common for all space directions. It is the additional unitary integrals in the QCD that "absorb" the space degrees of freedom.

Thus the induced QCD results into the low dimensional theory - one dimensional quantum mechanics for the partition function case and two dimensional theory for the scalar correlators. The latter fact is of special interest

because the two dimensional QCD models with a scalar matter in adjoint representation have received a recent attention due to a similarity they have with 4D gauge theory [8, 9]. The scalar degrees of freedom "compensate" the absence of transverse gluons in 2D models. From the other hand it is a commonplace that the linear potential resulting from the Coulomb interaction in 2D theories has no relation to 4D confinement. The dimensional reduction hints that such a relation might exist. Indeed, the interaction potential between the large mass matter particles can be extracted from the amplitude of their elastic  $2 \rightarrow 2$  scattering. It is the amplitude that allows for the two-dimensional reduction in the induced QCD. The Coulomb force originates from  $G_{\mu\nu}^2$  term in the effective lagrangian which appears after integration over the scalar field. However there is not the logarithmic divergency when  $D = 2$ , and the first term does not dominate in the loop expansion in which the higher terms are related to the presence of the non-Coulomb degrees of freedom.

The natural question here is in what extent these results are valid for the real, not induced, QCD. Unfortunately the naive picture supposing all the terms except the first one in the loop expansion to be suppressed by the large scalar mass  $M$  is inconsistent. The scalar loops result into the point-like gluon interaction only if the gluons' momenta are much smaller than the  $M$ . Therefore the induced QCD has to imply two typical scales to be equivalent to the gluodynamics – the ultraviolet cutoff  $\Lambda$  for the scalar particles and their mass  $M \ll \Lambda$  for the gluons. However the gluons' momenta circulating in the internal loops are restricted really by the  $\Lambda$  rather than the  $M$  value. That is why the expressions (14), (17) for the vacuum energy density can not be immediately applied to the QCD since the hard gluon loops contribution is not separated from them.

A possible way to connect the induced QCD with the real gluodynamics is to adopt a slightly different approach. One can start from the scalar QCD treating the scalars on an equal footing with the gluons so the lagrangian will encounter the scalars selfinteraction  $h\varphi^4$ . The constant  $h$  as well as the gluon coupling  $g$  are the bare constants which have to be taken to be functions of the regularization parameter  $\Lambda$ . Their values are determined by regularization procedure: the gluon and scalar dressed vertexes and the residues of the propagators poles are fixed at some external momentum scale  $\mu$  referred to as a normalization point. The position of the scalar propagator pole is fixed too as a physical scalar mass. If the bare constants are adjusted to keep all the fixed at the  $\mu$  point values when the ultraviolet cutoff  $\Lambda \rightarrow \infty$ , then according to the renormalization theory the Green functions will be finite. It is the dressed vertexes values at the normalization point rather than the bare constants that play the physical charges role in the theory.

The bare gluon coupling is zero in the induced QCD, therefore the total number of the bare parameters is insufficient to satisfy all the normalization

conditions. However one can lift one of them, say, not to fix the position of the scalar propagator poles using the bare scalar mass as a parameter to keep the gluon vertex value. Suppose the renormalization scheme can be carried out for all the vertexes and residues while the physical scalar mass turns out to be of the order of  $\Lambda$ . In this case the renormalized theory will be surely equivalent to the QCD. Even if the more complicated situation occurs it will be of interest to study what is the limiting theory. The obtained here results enable to investigate this problem in the framework of the two-dimensional scalar QCD.

There are several difficulties however to proceed in this manner. It is unclear if the triple gluon vertex allows to be reduced to the two-dimensional theory like the propagators do. The possible way to circumvent this obstacle is to determine the gluon physical coupling through the interaction with a large massive matter field, that is through the Coulomb potential. An other probably more serious difficulty is the absence of an exact solution of the massive two-dimensional QCD although there are numerical investigations of it [9].

Note in the conclusion that the induced QCD could be interesting in itself as a toy model even without the direct equivalence with the gluodynamics. Indeed it provides an example of the four-dimensional theory asymptotically free at the small distances (in some sense the asymptotic freedom in the induced QCD is even more strong than in the gluodynamics [10]) which exhibits a confinement-like behavior at the large ones.

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